

OPTIMAL EXPLICIT GUIDANCE FOR THREE-DIMENSIONAL LAUNCH TRAJECTORY†

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Abstract—The paper presents an exoatmospheric near-optimal explicit guidance algorithm for a satellite launch vehicle following a two- or three-dimensional trajectory. The unified steering law has been developed in a vector form using an equivalent gravitational field in an inertial coordinate frame. The resulting steering law follows a linear tangent law in a canted plane for a three-dimensional trajectory. The principle is based on the concepts of optimal transfer between non-coplanar orbits. The guidance parameters are determined by solving three simultaneous algebraic equations involving thrust integrals which are computed recursively. Two constant gravity related vectors are defined which account for change in velocity and position due to the spherical Earth. The solution is obtained by a differential correction method after finding the required partial derivatives analytically. A predictor corrector approach is suggested where the trajectory is predicted using the Encke's method. This also enables consideration of other effects such as oblateness. The simulation results for typical multistage launch vehicles show that the guidance algorithm is extremely accurate, robust and yet simple enough for on-board implementation. It can be used for a variety of missions.

1. INTRODUCTION

Closed-loop guidance is essential in upper stages of a launch vehicle to reach a specified orbit with minimum error. Many schemes have been suggested and used during the past three decades[1]. Most of the earlier schemes which are still in use were tailored to work under severe hardware limitations of on-board processors. With the present level of miniaturization and computational power, one can aim at evolving a more elegant, optimal and reliable closed-loop guidance scheme which gives higher accuracy, allows flexibility of use in a variety of missions, and permits fast computation of guidance parameters. This has been the objective of the present study.

The exoatmospheric guidance schemes can be classified under two major groups: velocity-correlated guidance, and explicit guidance schemes. The velocity-correlated guidance schemes[2] take the help of the nominal trajectory for a given mission for finding the guidance coefficients. Considerable pre-processing is involved in such schemes and the flexibility is limited. Explicit guidance methods are based essentially on solution of two-point boundary value problems. These schemes are being used widely in different forms. In most of the explicit guidance schemes available today, steering laws in pitch and yaw are derived separately after decoupling the equations in pitch and yaw planes[3–6]. This requires the simplifying assumption of yaw angle being quite small. Such decoupling leads to inaccuracy and loss of optimality particularly when the pitch and yaw

angles are not small. Use of numerical, rigorous calculus of variation methods[7–8] also has not been popular because of excessive computation load on on-board processor and a possibility of divergence. A simplified model is solved[9] semi-analytically with simultaneous numerical integration of thrust and gravity. However, this approach does not reduce the complexity of the calculus of variation problem significantly. In Ref.[10] the problem of high yaw manoeuvre is addressed and a bilinear tangent law for yaw and a linear tangent law for pitch are suggested separately.

This paper presents a near optimal explicit guidance scheme for launch vehicles which follow three-dimensional trajectories. The presence of landmass constraints and the location of the launch station along with the need for vehicle performance optimization sometimes necessitate the use of such trajectories, which demand large pitch and yaw manoeuvres. A linear tangent guidance law derived here in vector form for such missions overcomes the shortcomings of the existing approaches.

The approach adapts some concepts[11] of optimal transfer between orbits under the assumption of uniform gravitational field to find the plane of correction. The scheme also accounts for the effects of actual gravitational field but retains the simplicity of the flat-Earth (uniform gravity) approximation by defining two gravity related constant vectors corresponding to changes in position and velocity due to gravity. The effect of thrust is estimated in terms of changes in velocity and position. The parameters used in guidance logic, which influence the thrust effect, are modified so as to result in a trajectory allowing a satellite to be injected into the desired

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orbit. For predicting the trajectory, Encke's method from orbital mechanics is adapted. For faster computation, the reference trajectory is assumed to obey Keplerian relations and is computed analytically. This approach facilitates inclusion of oblateness effect and other small forces, resulting in a very accurate trajectory and helps in finding a near optimal solution for vehicle guidance.

The simulation results show that the algorithm is robust and can be used for launch with very high accuracy for a variety of missions.

2. DEVELOPMENT OF ALGORITHM

Development of the explicit closed-loop guidance algorithm requires the determination of the steering law in a parametric form. For this purpose first the effects of thrust and gravity are estimated separately in terms of changes in velocity and position. Then three guidance parameters and two gravity related vectors are determined which completely define the thrust steering programme and the thrust cut-off time to meet the desired injection conditions.

2.1. Coordinate systems

For the formulation of the algorithm, three right-handed rectangular coordinate systems have been used;

(i) The Launch Plane Inertial (LPI) Geocentric Coordinate frame $OX_1 Y_1 Z_1$ shown in Fig. 1(a) is the fundamental frame in which all vectors are expressed. It has OZ_1 pointing outward along the launch point frozen at take-off and OX_1 parallel to the launch azimuth.

(ii) Geocentric Orbital Coordinate frame $OX_0 Y_0 Z_0$ shown in Fig. 1(b) is defined with OX_0 pointing outward along the present position vector projected on the orbital plane, and OY_0 perpendicular to the desired orbital plane opposite to the angular momentum vector. The unit vector in OY_0 direction $\hat{I}_{y0}$ can be computed in terms of orbital elements as follows:

$$\hat{I}_{y0} = [A][-\sin i \sin \Omega, \sin i \cos \Omega, -\cos i]^T, \quad (1)$$

where $[A]$ transforms $\hat{I}_{y0}$ to LPI frame from an equatorial one.

(iii) In correction Plane Coordinate frame $OX_c Y_c Z_c$ shown in Fig. 2, OX_c and OY_c form the plane which contains r_{thr} . OY_c is along $-\mathbf{P}_r$, a costate vector related to change in position.

2.2. Optimal steering law

In inertial coordinate frame, the basic equations of motion of the launch vehicle for its flight in vacuum are:

$$\dot{\mathbf{v}} = F/m \mathbf{u} + \mathbf{g} \quad (2)$$

$$\dot{\mathbf{r}} = \mathbf{v} \quad (3)$$

where $\mathbf{v}$, $\mathbf{r}$ are the velocity and position vectors; F and m are instantaneous thrust and mass of the vehicle;

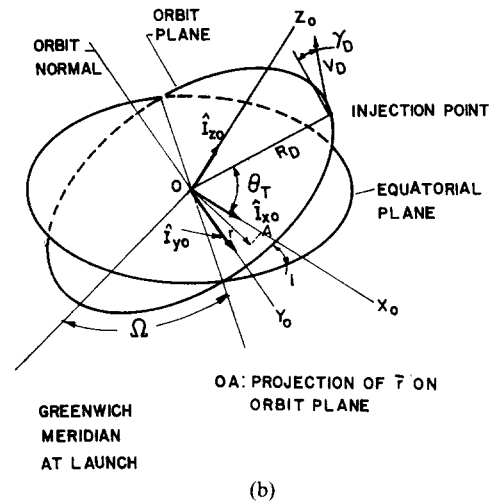
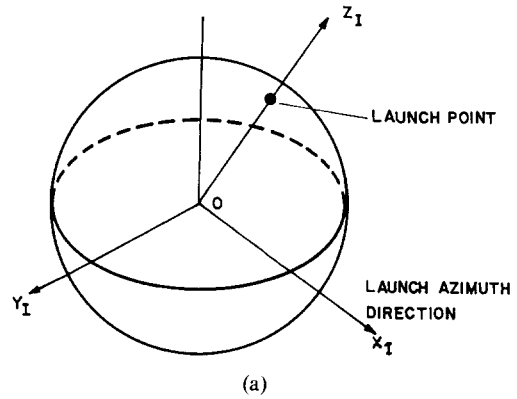


Fig. 1. (a) Launch Plane Inertial Coordinate frame. (b) Geocentric Orbital Coordinate frame.

$\mathbf{g}$ is the gravitational acceleration and $\mathbf{u}$ represents the unit thrust vector.

The optimization criterion chosen here is the minimization of time of burn for the rocket (or the last stage in the case of a multistage rocket) where thrust-time profile(s) (and the burning sequence of the different stages) is (are) specified. the problem

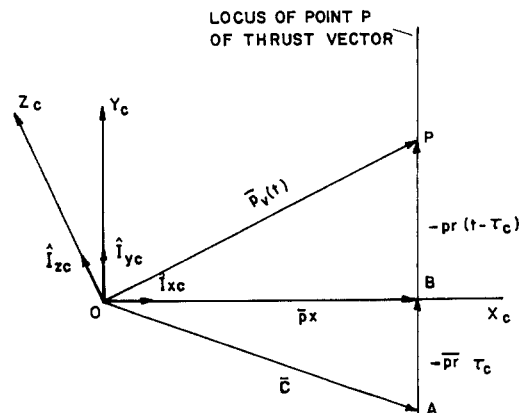


Fig. 2. Correction Plane Coordinate frame.

of minimum time transfer under the assumption of uniform gravity is well documented[11]. This can be used with appropriate modification to develop the guidance logic. The solution can be written in a vector form as:

$$\begin{aligned}\dot{\mathbf{p}}_v &= -\mathbf{p}_r, \\ \dot{\mathbf{p}}_r &= 0.\end{aligned}$$

Thus,

$$\mathbf{p}_v(t) = -\mathbf{p}_r t + \mathbf{c}, \quad (4)$$

where $\mathbf{p}_v$ is the Lawden's prime vector which is a costate vector associated with velocity and $\mathbf{p}_r$ is a constant vector defining the costate associated with position. Another constant vector is $\mathbf{c}$. Equation (4) shows that $\mathbf{p}_v(t)$ is contained in a plane generated by $\mathbf{c}$ and $\mathbf{p}_r$, as shown in Fig. 2, where OA represents $\mathbf{c}$. After resolving $\mathbf{c}$ into two components, along $\mathbf{p}_r$ and perpendicular to it, let the vector perpendicular to $\mathbf{p}_r$ be $\mathbf{p}_x$ along OB with the associated unit vector $\hat{\mathbf{I}}_{xc}$. Let $AB = -\mathbf{p}_r \tau_c$ where τ_c is a constant having unit of time. Hence,

$$\mathbf{c} = \hat{\mathbf{I}}_{xc} t_c + \mathbf{p}_r \tau_c.$$

On using this equation in eqn (4), one gets,

$$\mathbf{p}_v(t)/t_c = \hat{\mathbf{I}}_{xc} + \hat{\mathbf{I}}_{yc} p_\theta(t - \tau_c), \quad (5)$$

where $\hat{\mathbf{I}}_{yc}$ is the unit vector in the direction of $-\mathbf{p}_r$, and $p_\theta = |\mathbf{p}_r|/t_c$. The unit vector $\mathbf{u}$ in the direction of thrust can be obtained from eqn (5) by dividing the right-hand side by its magnitude, noting that $\hat{\mathbf{I}}_{xc}$ and $\hat{\mathbf{I}}_{yc}$ are orthogonal.

$$\mathbf{u}(t) = [\hat{\mathbf{I}}_{xc} + \hat{\mathbf{I}}_{yc} p_\theta(t - \tau_c)]/[1 + p_\theta^2(t - \tau_c)^2]^{1/2}. \quad (6)$$

This constitutes the optimum steering law. It can be seen from eqn (6) that the angle Ψ , which the thrust vector makes with $\hat{\mathbf{I}}_{xc}$ can be expressed as

$$\tan \Psi = p_\theta(t - \tau_c). \quad (7)$$

Hence, the tangent of the optimal steering angle follows a simple linear tangent law under uniform gravitational field.

2.3. Terminal constraints

The constraint relations at injection can be computed by first computing the desired injection state and then equating it with the integrated trajectory following a steering law up to thrust cut-off time.

For orbital injection, generally five out of six possible parameters are specified. They are R_D , V_D , γ_D and the desired orbital plane defined by unit vector normal to the orbit. The injection state then can be defined in terms of range angle θ_T and unit vectors, $\hat{\mathbf{I}}_{x0}$ and $\hat{\mathbf{I}}_{z0}$ (defined in Section 2.1) as:

$$\mathbf{v}_T = V_D[\hat{\mathbf{I}}_{x0} \sin(\gamma_D - \theta_T) + \hat{\mathbf{I}}_{z0} \cos(\gamma_D - \theta_T)], \quad (8)$$

$$\mathbf{r}_T = R_D[\hat{\mathbf{I}}_{x0} \cos(\theta_T) + \hat{\mathbf{I}}_{z0} \sin(\theta_T)]. \quad (9)$$

Alternatively, the injection constraints can be defined in terms of orbital elements, namely a , e , θ , i , Ω ,

where i and Ω define the orbit normal. ω is unconstrained. The corresponding values of R_D , V_D , and γ_D can be determined using the following relations:

$$R_D = a(1 - e^2)/(1 + e \cos \theta), \quad (10)$$

$$V_D = [\mu(2/R_D - 1/a)]^{1/2}, \quad (11)$$

$$\gamma_D = \cos^{-1}[\{\mu a(1 - e^2)\}^{1/2}/(R_D V_D)]. \quad (12)$$

This required state at injection leads to an equality constraint obtained by integrating the equations of motion up to T_{go} using $\mathbf{u}(t)$, which in turn is defined in terms of p_θ , τ_c , $\hat{\mathbf{I}}_{xc}$ and $\hat{\mathbf{I}}_{yc}$. It may be noted that there are only five constraints specified, hence injection can take place anywhere in the orbit. As the steering law is derived under the assumption of uniform gravitational field, it is valid for zero gravity field also, with appropriate change in the end constraints. The transformed end conditions allow one of the position components, along which total velocity constraint is met, to be unconstrained. It can be shown[12] that this corresponds to the direction of $\hat{\mathbf{I}}_{xc}$ (Fig. 2). Since the velocity constraint is met in the direction of $\hat{\mathbf{I}}_{xc}$, the net change in the velocity over the entire path in $\hat{\mathbf{I}}_{yc}$ direction should be zero. This relation is used to define τ_c . Since $\hat{\mathbf{I}}_{xc}$ and $\hat{\mathbf{I}}_{yc}$ are orthogonal, one is left with five independent parameters associated with $\mathbf{u}(t)$, (three defining $\hat{\mathbf{I}}_{xc}$ and $\hat{\mathbf{I}}_{yc}$, one each for p_θ and τ_c).

2.4. Effects of thrust

The changes in velocity and position due to thrust are estimated along with the corresponding changes due to gravity with assumed values or first approximation of the steering parameters. The values of the parameters are then modified in such a way that the resulting thrust direction programme guides the launch vehicle from the present state to the desired injection state.

The required change in velocity to be imparted by thrust is given by

$$\mathbf{v}_{thR} = \mathbf{v}_T - \mathbf{v}_0 - \mathbf{v}_g \quad (13)$$

where

$$\mathbf{v}_g = \int_0^{T_{go}} \mathbf{g}\{\mathbf{r}(t)\} dt. \quad (14)$$

The estimated change in velocity imparted by thrust for a specified set of steering parameters and T_{go} is given by

$$\mathbf{v}_{thA} = \int_0^{T_{go}} F(t)\mathbf{u}(t)/m(t) dt. \quad (15)$$

On substituting eqn (6) in to eqn (15) one obtains

$$\begin{aligned}\mathbf{v}_{thA} &= \hat{\mathbf{I}}_{xc} \int_0^{T_{go}} F(t)/[m(t)\{1 + p_\theta^2(t - \tau_c)^2\}^{1/2}] dt \\ &+ \hat{\mathbf{I}}_{yc} \int_0^{T_{go}} F(t)p_\theta(t - \tau_c)/[m(t) \\ &\times \{1 + p_\theta^2(t - \tau_c)^2\}^{1/2}].\end{aligned} \quad (16)$$

As explained in Section 2.3, the velocity constraint is to be met by the component of thrust in $\hat{\mathbf{i}}_{xc}$ direction only, therefore,

$$\mathbf{v}_{thA} = \mathbf{I}_{xc} \int_0^{T_{go}} F(t)/[m(t)\{1 + p_\theta^2(t - \tau_c)^2\}^{1/2}] dt \quad (17)$$

and

$$0 = \int_0^{T_{go}} F(t)p_\theta(t - \tau_c)/[m(t) \times \{1 + p_\theta^2(t - \tau_c)^2\}^{1/2}] dt. \quad (18)$$

Similarly, considering the position constraint, one gets the required change in position due to thrust:

$$\mathbf{r}_{thR} = \mathbf{r}_T - \mathbf{r}_o - \mathbf{v}_o T_{go} - \mathbf{r}_g, \quad (19)$$

where

$$\mathbf{r}_g = \int_0^{T_{go}} \int_0^t \mathbf{g}\{\mathbf{r}(s)\} ds dt. \quad (20)$$

The estimated change in position is given by:

$$\begin{aligned} \mathbf{r}_{thA} = & \hat{\mathbf{I}}_{xc} \int_0^{T_{go}} \int_0^t F(s)/[m(s) \\ & \times \{1 + p_\theta^2(s - \tau_c)^2\}^{1/2}] ds dt \\ & + \hat{\mathbf{I}}_{yc} \int_0^{T_{go}} \int_0^t F(s)p_\theta(s - \tau_c)/[m(s) \\ & \times \{1 + p_\theta^2(s - \tau_c)^2\}^{1/2}] ds dt. \end{aligned} \quad (21)$$

For the solution, the following constraints must be satisfied.

$$\mathbf{v}_{thR} = \mathbf{v}_{thA}, \quad (22)$$

$$\mathbf{r}_{thR} = \mathbf{r}_{thA}. \quad (23)$$

Equation (18) is solved to determine the value of τ_c . Since closed-form solutions of eqns (17), (18) and (21) are not available, some simplifications are made to make them integrable. For this the following relation is used.

$$[1 + p_\theta^2(t - \tau_c)^2]^{-1/2} = 1 - \frac{1}{2}p_\theta^2(t - \tau_c)^2 + \frac{3}{8}p_\theta^4(t - \tau_c)^4 - \dots \quad (24)$$

After selecting a finite number of terms of the above series and using eqn (24) in eqns (17), (18) and (21), one gets equations of the form:

$$\mathbf{I}_{Sn} = \int_0^{T_{go}} F(t)t^n/m(t) dt, \quad (25)$$

$$\mathbf{I}_{Dn} = \int_0^{T_{go}} \int_0^t F(s)s^n/m(s) ds dt. \quad (26)$$

These integrals can be solved analytically for constant thrust cases. For this scheme, the integrals are solved in a recursive way [12] making the approach computationally efficient. Equations (17), (18) and (21) can be written in algebraic form in terms of the integrals $\mathbf{I}_{Sn}$ and $\mathbf{I}_{Dn}$. For example, on using the first two terms of expansion from eqn (24), one gets the

following equations:

$$v_{thA} = I_{S0} - \frac{1}{2}p_\theta^2(I_{S2} - 2\tau_c I_{S1} + \tau_c^2 I_{S0}) \quad (27)$$

$$0 = (I_{S1} - \tau_c I_{S0}) - \frac{1}{2}p_\theta^2(I_{S3} - 3\tau_c I_{S2} + 3\tau_c^2 I_{S1} - \tau_c^3 I_{S0}) \quad (28)$$

$$\begin{aligned} r_{thA} = & [I_{D0} - \frac{1}{2}p_\theta^2(I_{D2} - 2\tau_c I_{D1} + \tau_c^2 I_{D0})]^2 \\ & + p_\theta^2[(I_{D1} - \tau_c I_{D0}) - \frac{1}{2}p_\theta^2(I_{D3} - 3\tau_c I_{D2} \\ & + 3\tau_c^2 I_{D1} - \tau_c^3 I_{D0})]^2. \end{aligned} \quad (29)$$

Although two terms of eqn (24) for expansion have been considered here, one can take as many number of terms as required for the desired accuracy.

2.5. Gravity effect modelling

The effect of gravity depends upon the trajectory followed by the launch vehicle, which in turn depends upon the value of the steering parameters. In conformity with the assumptions made in the derivation of the steering law, the gravity vector has to be taken to be uniform over the entire path. In order to represent the effects of the actual gravity on position and velocity vectors, two gravity related constant vectors are defined below:

$$\mathbf{g}_v = \int_0^{T_{go}} \mathbf{g}[\mathbf{r}(t)] dt / \left[\int_0^{T_{go}} dt \right] \quad (30)$$

and

$$\mathbf{g}_r = \int_0^{T_{go}} \int_0^t \mathbf{g}\{\mathbf{r}(s)\} ds dt / \left[\int_0^{T_{go}} \int_0^t ds dt \right]. \quad (31)$$

On using eqns (14) and (20), one gets,

$$\mathbf{g}_v = \mathbf{v}_g / T_{go} \quad (32)$$

and

$$\mathbf{g}_r = 2\mathbf{r}_g / T_{go}^2 \quad (33)$$

By choosing these two gravity related constant vectors, the simplicity of uniform gravity is maintained without sacrificing the accuracy of trajectory generation. For computing $\mathbf{v}_g$ and $\mathbf{r}_g$, Encke's approach of integrating the perturbations has been adapted to meet the fast computation requirement of the closed-loop guidance. The reference trajectory used corresponds to the Keplerian relations for two-body motion. The other forces acting on the vehicle are the thrust and a small force due to oblateness of the Earth. Encke's approach requires acceleration due to these perturbative forces alone to be integrated.

Let $\mathbf{r}_E$ be the position on the reference trajectory and $\mathbf{r}$ be the position on the actual trajectory which the vehicle follows when all forces act on it. The equations of motion can be written as:

$$\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3 + \Delta\ddot{\mathbf{r}}_{thrust} + \Delta\ddot{\mathbf{r}}_{oblate} \quad (34)$$

where $\Delta\ddot{\mathbf{r}}_{thrust}$ and $\Delta\ddot{\mathbf{r}}_{oblate}$ are the accelerations due to thrust and oblateness of the Earth respectively.

Let

$$\xi \triangleq \mathbf{r} - \mathbf{r}_E. \quad (35)$$

On using eqns (34) and (35), one gets,

$$\xi = \mu/r_E^3 [\mathbf{r} \{1 - (r_E/r)^3\} - \xi] + \Delta \mathbf{r}_{\text{thrust}} + \Delta \mathbf{r}_{\text{oblate}} \quad (36)$$

or

$$\xi = \mu/r_E^3 [(\mathbf{r}_E + \xi)f(q) - \xi] + \Delta \mathbf{r}_{\text{thrust}} + \Delta \mathbf{r}_{\text{oblate}} \quad (37)$$

where

$$f(q) = 1 - (1 + 2q)^{-3/2} \quad (38)$$

and

$$q = \{\xi \cdot (\mathbf{r}_E + \xi/2)\}/r_E^2. \quad (39)$$

Equation (37) can be written in a concise form as:

$$\xi = \Delta \mathbf{r}_{\text{Encke}} + \Delta \mathbf{r}_{\text{thrust}} + \Delta \mathbf{r}_{\text{oblate}}. \quad (40)$$

$\mathbf{r}_E$ and $\mathbf{v}_E$ can be related to the state at t_o by a transition matrix:

$$\begin{bmatrix} \mathbf{r}_E(t) \\ \mathbf{v}_E(t) \end{bmatrix} = \Phi(t, t_o) \begin{bmatrix} \mathbf{r}_E(t_o) \\ \mathbf{v}_E(t_o) \end{bmatrix}. \quad (41)$$

The solution for $\mathbf{r}_E(t)$ and $\mathbf{v}_E(t)$ are obtained analytically. Integration of $\Delta \mathbf{r}_{\text{thrust}}$ can be performed using Section 2.4. $\Delta \mathbf{r}_{\text{Encke}}$ and $\Delta \mathbf{r}_{\text{oblate}}$ need to be integrated numerically. Since these vectors are of very small magnitude as compared to that of $\mathbf{r}_E$, large step-size can be used for integration without sacrificing the desired accuracy. The expression for $\Delta \mathbf{r}_{\text{oblate}}$ is:

$$\begin{aligned} \Delta \mathbf{r}_{\text{oblate}} = & -\mu[3/2]J_{20}(R_e/r)^2 \\ & \times \{1 - 5(\mathbf{u}_z \cdot \mathbf{r})^2/r^2\} \mathbf{r}/r^3 - 3\mu J_{20}(\mathbf{R}_e/r)^2 \\ & \times (\mathbf{u}_z \cdot \mathbf{r})\mathbf{u}_z/r^3 \end{aligned} \quad (42)$$

where J_{20} is the second zonal harmonic term, $\mathbf{u}_z$ is the unit vector in the direction of the North Pole and R_e is the radius of the Earth. For numerical integration of $\Delta \mathbf{r}_{\text{Encke}}$ and $\Delta \mathbf{r}_{\text{oblate}}$, Simpson's scheme is found to be satisfactory.

Integration of eqn (40) along with eqn (41) provides the prediction at injection. With the above formulation, the predicted injection state is generally quite close to the desired one. This prediction helps in the modification of the guidance parameters to meet the injection constraints accurately. It may be noted that $\mathbf{g}_v$ and $\mathbf{g}_r$ computed from the predicted injection state converge to the correct value very fast as they represent the time averaged value over the entire path.

The vectors $\mathbf{g}_v$ and $\mathbf{g}_r$ are computed as follows:

$$\mathbf{g}_v = (\mathbf{v}'_T - \mathbf{v}_o - \mathbf{v}_{\text{thA}})/T_{\text{go}} \quad (43)$$

$$\mathbf{g}_r = 2(\mathbf{r}'_T - \mathbf{r}_o - \mathbf{v}T_{\text{go}} - \mathbf{r}_{\text{thA}})/T_{\text{go}}^2 \quad (44)$$

where $\mathbf{v}'_T$ and $\mathbf{r}'_T$ are the predicted velocity and position vectors at the injection using Encke's method.

2.6. Guidance parameters

The parameters $\hat{\mathbf{I}}_{\text{xc}}, \hat{\mathbf{I}}_{\text{yc}}, p_\theta, \tau_c$ define the steering law [eqn (6)] completely. These are calculated after deter-

mining the parameters T_{go}, p_θ and θ_T which are found by iteration after converting the vector eqns (22) and (23) into the corresponding algebraic scalar equations and applying a differential correction approach. The scalar algebraic equations corresponding to eqns (22) and (23) are:

$$F1 = \mathbf{v}_{\text{thR}} \cdot \mathbf{v}_{\text{thR}} - \{(\mathbf{v}_{\text{thA}} \cdot \hat{\mathbf{I}}_{\text{xc}})^2\} \quad (45)$$

$$F2 = \mathbf{r}_{\text{thR}} \cdot \mathbf{r}_{\text{thR}} - (\mathbf{r}_{\text{thA}} \cdot \hat{\mathbf{I}}_{\text{xc}})^2 - (\mathbf{r}_{\text{thA}} \cdot \hat{\mathbf{I}}_{\text{yc}})^2 \quad (46)$$

$$F3 = \mathbf{v}_{\text{thR}} \cdot \mathbf{r}_{\text{thR}} - \mathbf{v}_{\text{thA}} \cdot \mathbf{r}_{\text{thA}} \quad (47)$$

$$F4 = \mathbf{v}_{\text{thA}} \cdot \hat{\mathbf{I}}_{\text{yc}}. \quad (48)$$

τ_c is determined using eqn (48). At the correct value of T_{go}, p_θ and θ_T , $F1, F2, F3$ should be simultaneously zero. Any small change in their value results in an error in $F1, F2, F3$ according to the following relations:

$$\begin{aligned} \partial F1/\partial T_{\text{go}} dT_{\text{go}} + \partial F1/\partial p_\theta dp_\theta \\ + \partial F1/\partial \theta_T d\theta_T = dF1, \end{aligned} \quad (49)$$

$$\begin{aligned} \partial F2/\partial T_{\text{go}} dT_{\text{go}} + \partial F2/\partial p_\theta dp_\theta \\ + \partial F2/\partial \theta_T d\theta_T = dF2, \end{aligned} \quad (50)$$

$$\begin{aligned} \partial F3/\partial T_{\text{go}} dT_{\text{go}} + \partial F3/\partial p_\theta dp_\theta \\ + \partial F3/\partial \theta_T d\theta_T = dF3, \end{aligned} \quad (51)$$

where the partial derivative can be found analytically by differentiating the right-hand side of eqns (45), (46) and (47) with respect to T_{go}, p_θ and θ_T . From these linearly independent algebraic equations, the corrections needed in the estimates of T_{go}, p_θ and θ_T to make Fi ($i = 1, 2, 3$) zero can be found from the errors in Fi by matrix inversion. The scheme exhibits good convergence. Generally, it takes only one or two iterations for the solution except at the start of guidance where it may take up to 5–6 iterations depending upon errors in initial estimates of guidance parameters. The convergence of T_{go} [i.e. $|dT_{\text{go}}/T_{\text{go}}| < \epsilon$ (a small number)] ensures the convergence of p_θ and θ_T as well.

With these, the value of $\mathbf{v}_{\text{thR}}$ and $\mathbf{r}_{\text{thR}}$ can be found by using eqns (13) and (19) respectively. $\hat{\mathbf{I}}_{\text{xc}}$ and $\hat{\mathbf{I}}_{\text{yc}}$ are then determined as follows:

$$\hat{\mathbf{I}}_{\text{xc}} = \text{unit}(\mathbf{v}_{\text{thR}}) \quad (52)$$

$$\hat{\mathbf{I}}_{\text{yc}} = \text{unit}(\hat{\mathbf{I}}_{\text{xc}} \times \mathbf{r}_{\text{thR}}) \times \hat{\mathbf{I}}_{\text{xc}}. \quad (53)$$

Thus, all parameters in eqn (6) defining the steering law are fully determined.

2.7. Terminal processing

In the vicinity of the injection point, most of the guidance schemes tend to instability. This originates from the terminal constraint which requires zero injection error, and which may demand infinite control gains and very fast turn over rates close to injection. Therefore, one generally switches over to another guidance mode.

In the present case, the position constraint imposed by eqns (46) and (47) are relaxed once T_{go} is less than

$T_{\min}$, a predefined time based on simulation. Thereafter, only one equation, namely eqn (45) is solved for T_{go} , keeping the injection state and p_{θ} the same as determined earlier.

2.8. Multistage rocket

For a multistage rocket the guidance algorithm needs some modification. The computation of $\mathbf{v}_{thA}$ and $\mathbf{r}_{thA}$ are different, as one has to compute the total changes in velocity and position imparted by the current and remaining stages. Also, while solving the double integrals, one should include the contributions by the corresponding single integrals at the end of the previous stages. Other computations are identical to the ones corresponding to the single stage.

As it is simpler to compute the thrust integrals of eqns (25) and (26) between the limits of zero and a definite time, the value of τ_c has been correspondingly modified for the remaining stages. Thus

$$\tau_{c_{i+j}} = \tau_{c_i} - \Delta t_{i+j} \quad (54)$$

where τ_{c_i} is τ_c for the current stage and Δt_{i+j} is the time difference between ignition time of $(i+j)$ stage and the current time. This way τ_c for all the remaining stages can be defined.

The expression for $\mathbf{v}_{thA}$ can be written as,

$$\mathbf{v}_{thA} = \mathbf{I}_{xc} \sum_{i=\text{current stage}}^n \int_0^{T_i} F_i/[m_i \times \{1 + p_{\theta}^2(t - \tau_{c_i})^2\}^{1/2}] dt \quad (55)$$

where T_i is the remaining burn time for the i th stage. τ_c for the current stage is computed using the following relation:

$$0 = \sum_{i=\text{current stage}}^n \int_0^{T_i} F_i p_{\theta}(t - \tau_{c_i})/[m_i \times \{1 + p_{\theta}^2(t - \tau_{c_i})^2\}^{1/2}] dt \quad (56)$$

τ_{c_i} for all stages are expressed in terms of τ_c for the current stage, which is then found using eqn (56).

The expression for $\mathbf{r}_{thA}$ can also be written in a similar manner on using the following relation:

$$\begin{aligned} \mathbf{r}_{thA} = & \mathbf{I}_{xc} \sum_{i=\text{current stage}}^n \left[\int_0^{T_i} \int_0^t F_i(s)/[m(s) \times \{1 + p_{\theta}^2(s - \tau_{c_i})^2\}^{1/2}] ds dt \right. \\ & + t_{c_i} \int_0^{T_i} F_i(t)/[m(t) \{1 + p_{\theta}^2(t - \tau_{c_i})^2\}^{1/2}] dt \left. \right] \\ & + \mathbf{I}_{xc} \sum_{i=\text{current stage}}^n \left[\int_0^{T_i} \int_0^t F_i(s) p_{\theta}(s - \tau_{c_i})/[m(s) \times \{1 + p_{\theta}^2(s - \tau_{c_i})^2\}^{1/2}] ds dt \right. \\ & + t_{c_i} \int_0^{T_i} F_i(t)/[m(t) \{1 + p_{\theta}^2(t - \tau_{c_i})^2\}^{1/2}] dt \left. \right] \quad (57) \end{aligned}$$

where t_{c_i} is the remaining time for the thrust cut-off after the end of the i th stage.

For computation of gravity and prediction of state at the expected injection, Encke's method is used. The trajectory is corrected at the end of each stage using the $\mathbf{v}_{thA}$ and $\mathbf{r}_{thA}$ corresponding to the particular stage as given in Section 2.5.

The guidance parameters are determined in a manner identical to the one given in Section 2.6.

3. SIMULATION RESULTS

Performance studies are carried out for a representative multi-stage launch vehicle which employs a 3-dimensional trajectory due to land-mass constraint, for launching a satellite into a sun-synchronous orbit of altitude 900 km. The closed loop guidance is operational for nearly 1000 s including a long coast before the final stage ignition. In total, it covers a range angle of nearly 40° , requiring nearly 120° of manoeuvre in pitch and nearly 20° manoeuvre in yaw. The update of guidance parameter is done for every 5 s and the steering angle is computed based on determined guidance parameters at every 0.5 s. The position constraint is relaxed if $T_{go} < 5$ s. For estimating the thrust effect, the first two terms of series expansion are retained.

The salient features of the results are shown in Table 1. The results for a nominal flight shows very good accuracy at injection. The error in semi-major axis is < 10 cm and that in eccentricity is around 10^{-7} . The position error and errors in apogee and perigee are all less than 1 m. Figure 3 shows the pitch and yaw angles for the duration where guidance was operational. During the coast, steering is ineffective, hence, these angles have not been shown.

This high accuracy has been possible due to accurate prediction of the trajectory. The prediction results are shown in Table 2. It shows that even with no update of guidance parameters for the last 150 s, the error in semi-major axis is < 16 m and error in eccentricity is around 10^{-5} . The prediction error has been found by first computing the injection state by numerical integration using the vehicle's position and velocity at some previous time with the steering parameters computed at that time and then comparing the injection condition with the desired one.

Many cases of perturbations have been simulated to test the robustness of the algorithm. Typical results of the following cases have been shown in Table 1.

Case 1. This case has been simulated to test the explicitness of the algorithm by varying the initial position and velocity at the initiation of the guidance. For this, each component of the position vector is changed by 50 km and each component of velocity vector is changed by 50 m/s. This does not affect adversely the initial convergence or the final accuracy of injection. The change in burn-time is 17.8 s.

Case 2. While considering the variation in the system parameters, the most dominant effect of per-

Table 1. Simulation of injection error—Sun-synchronous orbit

Case No.	Description	Change in burning-time (s)	Error in eccentricity	Abs. error in semi-major axis (m)	Abs. error in orbit normal (deg)	Abs. error in inj. position (m)	Abs. error in inj. vel. (m/s)
	Nominal performance with no perturbation	—	1.24×10^{-7}	0.086	7.54×10^{-6}	0.955	2.88×10^{-7}
1.	Change in initial positions: 50 km in each components Change in initial velocity: 50 m/s in each component	17.8	1.05×10^{-6}	0.10	6.0×10^{-5}	7.63	1.61×10^{-5}
2.	Change in specific impulse: penultimate stage, 1%; final stage, 0.75%	6.36	1.74×10^{-5}	0.33	9.9×10^{-4}	4.39	0.133
3.	Change in specific impulse for all stages, 5%	57.12	1.33×10^{-4}	116.2	7.54×10^{-3}	297.04	0.675
4.	Change in structure weight: penultimate stage, 3%; final stage, 1.5%	8.94	3.6×10^{-5}	10.05	2.05×10^{-3}	266.17	5.52×10^{-3}
5.	All perturbations included: Change in specific impulse: same as case 4 change in propellant wt: penultimate stage, 0.5%	19.1	7.86×10^{-5}	52.71	4.48×10^{-3}	577.24	1.63×10^{-2}

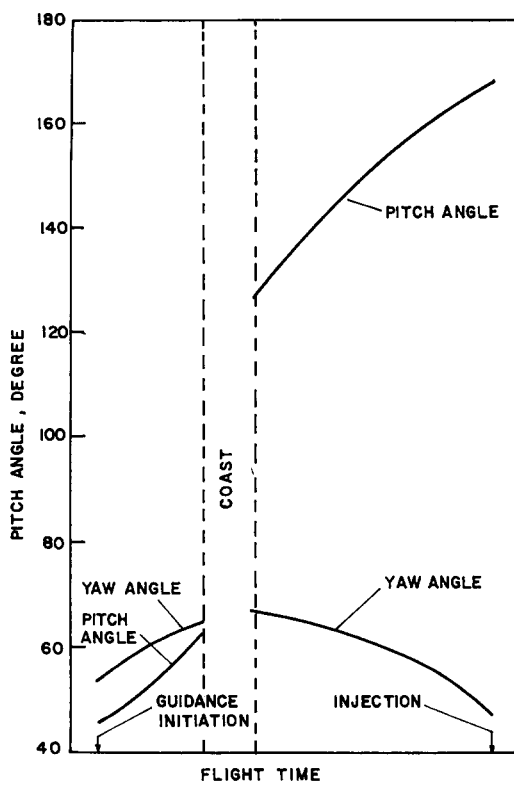


Fig. 3. Pitch and yaw angles for a Sun-synchronous mission.

turbation is due to variation of specific impulse of the propellant used in the motor. This case is simulated to test the robustness of the algorithm under 3σ variation of the specific impulse, which is taken as 1% in the penultimate stage and 0.75% in the final stage. The error in semi-major axis is 0.33 m and error in eccentricity is 1.7×10^{-5} . The change in burn-time is small.

Case 3. This case also has been simulated with a reduction in specific impulse in all the stages by 5% as compared to the nominal one. This high value of specific impulse variation has been considered to highlight that the algorithm is very robust. The convergence takes nearly 3 iterations most of the time. The error in semi-major axis is around 120 m and eccentricity error is 1.3×10^{-4} , which is well within the specification of the injection accuracy of such missions. The penalty is significant increase in the burn-time of the last stage.

Case 4. The source of perturbation other than the variation in the mass of the vehicle. This has been simulated by considering 1.5% of higher structural weight in the final stage and 3% of higher structural weight in the penultimate stage. The resulting error in semi-major axis is around 10 m, and error in eccentricity is 3.6×10^{-5} . This result shows that the algorithm is quite robust.

Case 5. The robustness of the algorithm has been further tested under the extreme perturbation case

Table 2. Prediction accuracy

Parameter for which error is computed	Prediction duration		
	150 s	250 s	400 s
Semi-major axis (m)	15.7	-63.3	-602.8
Eccentricity	2.27×10^{-5}	3.79×10^{-5}	1.53×10^{-4}
Position X (m)	-23.27	-41.37	-9.26
Position Y (m)	125.57	214.40	456.95
Position Z (m)	103.69	172.63	582.96
Velocity $\dot{X}$ (m/s)	-2.53×10^{-3}	-2.89×10^{-3}	-2.60×10^{-2}
Velocity $\dot{Y}$ (m/s)	1.59×10^{-3}	3.30×10^{-2}	5.33×10^{-1}
Velocity $\dot{Z}$ (m/s)	-6.96×10^{-3}	2.29×10^{-2}	2.54×10^{-1}

Table 3. Case study of geostationary transfer orbit injection

Error in perigee height	13.01 m
Error in apogee height	-17.15 m
Error in semi-major axis	-2.07 m
Error in eccentricity	6.4×10^{-7}
Error in injection position:	
Component X	-0.89 m
Component Y	0 m
Component Z	0.16 m
Error in injection velocity:	
Component $\dot{X}$	-1.85×10^{-6} m/s
Component $\dot{Y}$	0 m/s
Component $\dot{Z}$	1.35×10^{-6} m/s

where perturbations due to variation in specific impulse, structure weight and propellant weight all have been considered simultaneously as follows: increase in structure weight: 3% in the penultimate stage, 1.5% in the final stage; decrease in specific impulse: 1% in the penultimate stage, 0.75% in the final stage; decrease in propellant weight: 0.5% in the penultimate stage. The final error in semi-major axis is round 50 m and error in eccentricity is around 8×10^{-5} with 19.1 s increase in burn-time.

All these results prove that the algorithm is robust under various perturbations possible during flight and is capable of injecting the satellite into the desired orbit very accurately.

Performance of the scheme has also been tested for a geostationary transfer orbit mission which employs a planar launch trajectory. The launch vehicle for which simulation studies have been made has three stages and closed-loop guidance is operational in the last two stages. The result in Table 3 shows that the accuracy of the injection is good. The error in semi-major axis is only 2 m and error in velocity is

less than 2×10^{-6} m/s. The position error at injection is less than 1 m. Figure 4 shows the pitch angle and pitch rate during the period when guidance is operative.

4. CONCLUSION

An accurate, efficient and truly explicit closed-loop guidance algorithm has been formulated in a vector form using Launch Plane Inertial Coordinate frame which works well for a launch vehicle following 2- and 3-dimensional trajectory even with relatively large range angle and long operation time.

The algorithm is quite flexible in its computational load demand. The number of terms considered for thrust effect estimation can be properly chosen to suit the on-board computational power and a trade-off can be made between prediction accuracy and computational load. The simultaneous computation of guidance parameters avoids the need of separate modules to compute time-to-go, range angle and rate of turn of thrust angle and ensures fast and true convergence of guidance parameters. The accuracy of predicted trajectory using Encke's method allows the guidance parameter update cycle to be longer and at the same time helps in the getting a steering solution which results in an optimal and accurate guidance scheme.

The inertial navigation coordinate frame used in the formulation avoids all coordinate transformation requiring computations of trigonometrical functions.

With its high accuracy and flexibility, the algorithm can be used for a variety of missions. Here its applicability in the case of a 3-dimensional launch to reach a Sun-synchronous orbit and a 2-dimensional launch to reach a transfer orbit for geostationary missions is demonstrated.

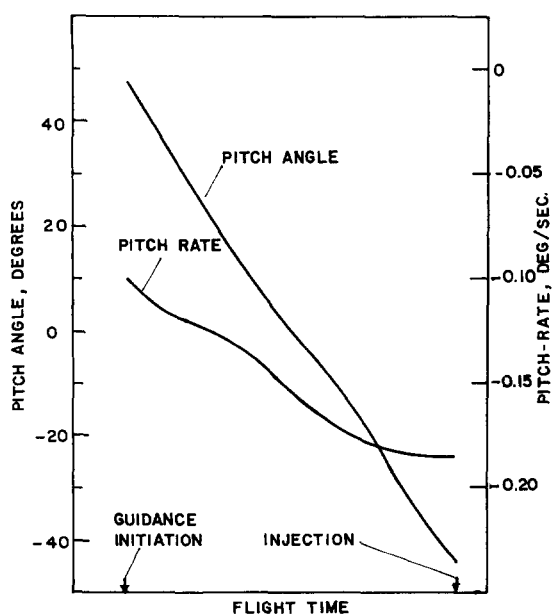


Fig. 4. Pitch and pitch rate for GTO mission.

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APPENDIX

Nomenclature

$a, e, i, \omega, \Omega, \theta$ = semi-major axis, eccentricity, inclination, argument of perigee, longitude of the ascending node and true anomaly of the resulting orbit at injection
 F, m = instantaneous thrust and mass of the vehicle

$\mathbf{i}_{qc}, \mathbf{i}_{qo} (q = x, y, z)$ = unit vectors in correction and orbital coordinate frames
 $p_{\dot{\theta}}$ = a guidance parameter corresponding to rate of change of thrust angle
 $\mathbf{r}_g, \mathbf{v}_g$ = change in position and velocity due to gravity computed from the present time until injection
 $\mathbf{r}_o, \mathbf{v}_o$ = present position and velocity obtained from navigation system
 $\mathbf{r}_T, \mathbf{v}_T$ = desired position and velocity at injection
 $\mathbf{r}_{thA}, \mathbf{r}_{thR}$ = change in position estimated and required to be imparted by thrust
 R_D, V_D, γ_D = desired distance, velocity and flight path angle at injection
 T_{go} = time-to-go, the thrust cut-off time measured from the present time
 T_{min} = the value of T_{go} after which position constraint is relaxed
 $\mathbf{v}_{thA}, \mathbf{v}_{thR}$ = change in velocity estimated and required to be imparted by thrust
 θ_T = range angle in the orbital plane measured between the projection of the present position vector and estimated injection location
 τ_c = time parameter in the steering law
 μ = Earth's gravitational constraint.